Baltic Way 2020

Version: *English*

November 14th, 2020

Time allowed: 4.5 hours.
During the first 30 minutes, questions may be asked.
Tools for writing and drawing are the only ones allowed for problem solving.

Problem 1. Let $a_0 > 0$ be a real number, and let

$$a_n = \frac{a_{n-1}}{\sqrt{1 + 2020 \cdot a_{n-1}^2}}, \quad \text{for } n = 1, 2, \ldots, 2020.$$

Show that $a_{2020} < \frac{1}{2020}$.

Problem 2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a\sqrt{c^2+1}} + \frac{1}{b\sqrt{a^2+1}} + \frac{1}{c\sqrt{b^2+1}} > 2.$$

Problem 3. A real sequence $(a_n)_{n=0}^{\infty}$ is defined recursively by $a_0 = 2$ and the recursion formula

$$a_n = \begin{cases} a_{n-1}^2 & \text{if } a_{n-1} < \sqrt{3} \\ \dfrac{a_{n-1}^2}{3} & \text{if } a_{n-1} \geqslant \sqrt{3}. \end{cases}$$

Another real sequence $(b_n)_{n=1}^{\infty}$ is defined in terms of the first by the formula

$$b_n = \begin{cases} 0 & \text{if } a_{n-1} < \sqrt{3} \\ \dfrac{1}{2^n} & \text{if } a_{n-1} \geqslant \sqrt{3}, \end{cases}$$

valid for each $n \geqslant 1$. Prove that

$$b_1 + b_2 + \cdots + b_{2020} < \frac{2}{3}.$$

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$f(f(x) + x + y) = f(x + y) + yf(y)$$

for all real numbers x, y.

Problem 5. Find all real numbers x, y, z so that

$$x^2y + y^2z + z^2 = 0$$
$$z^3 + z^2y + zy^3 + x^2y = \frac{1}{4}(x^4 + y^4)$$

Problem 6. Let $n > 2$ be a given positive integer. There are n guests at Georg's bachelor party and each guest is friends with at least one other guest. Georg organizes a party game among the guests. Each guest receives a jug of water such that there are no two guests with the same amount of water in their jugs. All guests now proceed simultaneously as follows. Every guest takes one cup for each of his friends at the party and distributes all the water from his jug evenly in the cups. He then passes a cup to each of his friends. Each guest having received a cup of water from each of his friends pours the water he has received into his jug. What is the smallest possible number of guests that do not have the same amount of water as they started with?

Problem 7. A mason has bricks with dimensions $2 \times 5 \times 8$ and other bricks with dimensions $2 \times 3 \times 7$. She also has a box with dimensions $10 \times 11 \times 14$. The bricks and the box are all rectangular parallelepipeds. The mason wants to pack bricks into the box filling its entire volume and with no bricks sticking out. Find all possible values of the total number of bricks that she can pack.

Problem 8. Let n be a given positive integer. A restaurant offers a choice of n starters, n main dishes, n desserts and n wines. A merry company dines at the restaurant, with each guest choosing a starter, a main dish, a dessert and a wine. No two people place exactly the same order. It turns out that there is no collection of n guests such that their orders coincide in three of these aspects, but in the fourth one they all differ. (For example, there are no n people that order exactly the same three courses of food, but n different wines.) What is the maximal number of guests?

Problem 9. Each vertex v and each edge e of a graph G are assigned numbers $f(v) \in \{1,2\}$ and $f(e) \in \{1,2,3\}$, respectively. Let $S(v)$ be the sum of numbers assigned to the edges incident to v plus the number $f(v)$. We say that an assignment f is *cool* if $S(u) \neq S(v)$ for every pair (u,v) of adjacent (i.e. connected by an edge) vertices in G. Prove that for every graph there exists a cool assignment.

Problem 10. Alice and Bob are playing hide and seek. Initially, Bob chooses a secret fixed point B in the unit square. Then Alice chooses a sequence of points $P_0, P_1, \ldots, P_N$ in the plane. After choosing P_k (but before choosing P_{k+1}) for $k \geqslant 1$, Bob tells "warmer" if P_k is closer to B than P_{k-1}, otherwise he says "colder". After Alice has chosen P_N and heard Bob's answer, Alice chooses a final point A. Alice wins if the distance AB is at most $\frac{1}{2020}$, otherwise Bob wins. Show that if $N = 18$, Alice cannot guarantee a win.

Problem 11. Let ABC be a triangle with $AB > AC$. The internal angle bisector of $\angle BAC$ intersects the side BC at D. The circles with diameters BD and CD intersect the circumcircle of $\triangle ABC$ a second time at $P \neq B$ and $Q \neq C$, respectively. The lines PQ and BC intersect at X. Prove that AX is tangent to the circumcircle of $\triangle ABC$.

Problem 12. Let ABC be a triangle with circumcircle ω. The internal angle bisectors of $\angle ABC$ and $\angle ACB$ intersect ω at $X \neq B$ and $Y \neq C$, respectively. Let K be a point on CX such that $\angle KAC = 90°$. Similarly, let L be a point on BY such that $\angle LAB = 90°$. Let S be the midpoint of arc CAB of ω. Prove that $SK = SL$.

Problem 13. Let ABC be an acute triangle with circumcircle ω. Let ℓ be the tangent line to ω at A. Let X and Y be the projections of B onto lines ℓ and AC, respectively. Let H be the orthocenter of BXY. Let CH intersect ℓ at D. Prove that BA bisects angle CBD.

Problem 14. An acute triangle ABC is given and let H be its orthocenter. Let ω be the circle through B, C and H, and let Γ be the circle with diameter AH. Let $X \neq H$ be the other intersection point of ω and Γ, and let γ be the reflection of Γ over AX.

Suppose γ and ω intersect again at $Y \neq X$, and line AH and ω intersect again at $Z \neq H$. Show that the circle through A, Y, Z passes through the midpoint of segment BC.

Problem 15. On a plane, Bob chooses 3 points A_0, B_0, C_0 (not necessarily distinct) such that $A_0B_0 + B_0C_0 + C_0A_0 = 1$. Then he chooses points A_1, B_1, C_1 (not necessarily distinct) in such a way that $A_1B_1 = A_0B_0$ and $B_1C_1 = B_0C_0$. Next he chooses points A_2, B_2, C_2 as a permutation of points A_1, B_1, C_1. Finally, Bob chooses points A_3, B_3, C_3 (not necessarily distinct) in such a way that $A_3B_3 = A_2B_2$ and $B_3C_3 = B_2C_2$. What are the smallest and the greatest possible values of $A_3B_3 + B_3C_3 + C_3A_3$ Bob can obtain?

Problem 16. Richard and Kaarel are taking turns to choose numbers from the set $\{1, \ldots, p-1\}$ where $p > 3$ is a prime. Richard is the first one to choose. A number which has been chosen by one of the players cannot be chosen again by either of the players. Every number chosen by Richard is multiplied with the next number chosen by Kaarel. Kaarel wins the game if at any moment after his turn the sum of all of the products calculated so far is divisible by p. Richard wins if this does not happen, i.e. the players run out of numbers before any of the sums is divisible by p. Can either of the players guarantee their victory regardless of their opponent's moves and if so, which one?

Problem 17. For a prime number p and a positive integer n, denote by $f(p,n)$ the largest integer k such that $p^k \mid n!$. Let p be a given prime number and let m and c be given positive integers. Prove that there exist infinitely many positive integers n such that $f(p,n) \equiv c \pmod{m}$.

Problem 18. Let $n \geqslant 1$ be a positive integer. We say that an integer k is a *fan* of n if $0 \leqslant k \leqslant n-1$ and there exist integers $x, y, z \in \mathbb{Z}$ such that

$$\begin{aligned} x^2 + y^2 + z^2 &\equiv 0 \pmod{n}; \\ xyz &\equiv k \pmod{n}. \end{aligned}$$

Let $f(n)$ be the number of fans of n. Determine $f(2020)$.

Problem 19. Denote by $d(n)$ the number of positive divisors of a positive integer n. Prove that there are infinitely many positive integers n such that $\lfloor \sqrt{3} \cdot d(n) \rfloor$ divides n.

Problem 20. Let A and B be sets of positive integers with $|A| \geq 2$ and $|B| \geq 2$. Let S be a set consisting of $|A| + |B| - 1$ numbers of the form ab where $a \in A$ and $b \in B$. Prove that there exist pairwise distinct $x, y, z \in S$ such that x is a divisor of yz.